**Ch. 1 Lenses and ray tracing ideas**

P’

P

H’

H

F

F’

‘

**Fig. 1**

Gauss’s Lens law from geometry:

From observing magnifying glasses we can postulate:

* Assumption of planes H and H’ where image has same height as object i.e. unit magnification.
* Point F and F’ where parallel rays must converge to single point.

Equations derived from purely geometrical arguments:

|  |  |  |
| --- | --- | --- |
|  |  | (1) |
|  |  | (2) |

From similar triangles we can observe from Fig. 1 that,  
Substituting (2) into (1) we find,

|  |  |  |
| --- | --- | --- |
|  |  | (3) |

If we define image magnification as, then also using (1) and (2),

|  |  |  |
| --- | --- | --- |
|  |  | (4) |

And also,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (5) |

Further from (1) and (5) we can show that,

|  |  |  |
| --- | --- | --- |
|  |  | (6) |

Therefore, we derive Guass’s lens equation from (6) which is ultimately derived from the two postulates and simple geometrical arguments.

|  |  |  |
| --- | --- | --- |
|  |  | (7) |

Ray tracing to Gaussian (abcd) constants to system matrices

C

**Fig. 2**

For the paraxial approximation (i.e. small angles such that rays are nearly parallel but not quite) we use the following approximations:

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  |  |

From Fig. 2 and the paraxial approximation:

|  |  |  |
| --- | --- | --- |
|  |  | (8) |

From Snell’s law applied to surface and the paraxial approximation (from now on we use this approximation and shall not refer to it):

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (9) |

(Snell’s law accounts for the fact that rays travel along the path which takes the least time accounting for the different velocities of light in different media).

From Fig. 2,

|  |  |  |
| --- | --- | --- |
|  |  | (10) |
|  |  | (11) |

Substituting (10) and (11) into (9) and using (8) gives,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  |  | (12) |

Where, is the refracting power of surface .

Also we have defined,

|  |  |  |
| --- | --- | --- |
|  |  | (13) |

Writing (12) and (13) in matrix form we have,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (14) |

Where, is the refraction matrix for surface . The subscript (from right to left simply identifies that the transformation occurs from surface and remains at surface .

For the translation from surface to surface we have,

|  |  |  |
| --- | --- | --- |
|  | (i.e angles don’t change) | (15) |
|  |  | (16) |

Writing (15) and (16) in matrix form we have,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (17) |

Where is the translation matrix from surface to surface . Notice the order of subscripts from right to left indicates translation occurs from one surface to another.

Combining, (14) and (17) we can obtain,

|  |  |  |
| --- | --- | --- |
|  |  |  |

We can define in general the following matrix blocks for refraction and translation which we can apply for any paths and surfaces involving a combination of translation and refractions,

|  |  |  |
| --- | --- | --- |
|  | (refraction transformation) | (18) |
|  | (translation transformation) | (19) |

Where, is the refracting power of the surface . Note: represents the translation matrix from surface to surface .

Including refraction at the system matrix equation for the lens is,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (20) |

Where the system matrix,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (21) |

Where it can be found after multiplying out the matrices,

|  |  |  |
| --- | --- | --- |
|  |  | (22) |
|  |  | (23) |
|  |  | (24) |
|  |  | (25) |

Where - are referred to as the Gaussian constants. Note one of the equations (22)-(25) are redundant since we also have,

|  |  |  |
| --- | --- | --- |
|  |  | (26) |

Ray tracing from object to image

P’

P

H’

H

F

F’

‘

**Fig. 3 Labelling distances from cardinal planes P, P’, H, H’, F, and F’ to lens.**

We define displacements to the left of to be negative (in fact labels representing displacements to the left of lens in Fig. 3 should be represented by their absolute values or the negative of the variables specified).

Therefore from the translation transformation matrix for the paraxial approximation (19) we can find that:

|  |  |  |
| --- | --- | --- |
|  | (translation from object to surface ) |  |
|  | |  |  |  | | --- | --- | --- | |  | (translation from surface to image) |  | |  |

Therefore, the complete matrix equation connecting object to image is,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |
|  | |  |  |  | | --- | --- | --- | |  |  |  | |  |
|  |  | (27) |

Where,

Since, , that would imply for non-blurry image that the magnification cannot depend on the angle of the ray leaving the object . Thus, by observation of (27) the (2,1) element of the system matrix must be null giving,

|  |  |  |
| --- | --- | --- |
|  |  | (28) |

Furthermore, by observation of (27) and also that , we must also have,

|  |  |  |
| --- | --- | --- |
|  |  | (29) |

Thus,

|  |  |  |
| --- | --- | --- |
|  |  | (30) |

From (28) we can show that the following must hold for non-blurry image,

|  |  |  |
| --- | --- | --- |
|  |  | (31) |

Attaching meaning to Gaussian constants

For unit magnification, , the object is at and the image is at . Therefore, we have from (29),

|  |  |  |
| --- | --- | --- |
|  |  | (32) |
|  |  |  |

And,

|  |  |  |
| --- | --- | --- |
|  |  | (33) |
|  |  |  |

Also, image is of zero height, , iff and . Therefore from (29) we have

|  |  |  |
| --- | --- | --- |
|  |  | (34) |
|  |  |  |

Also, if object is at focal point i.e. , we have and . Therefore,

|  |  |  |
| --- | --- | --- |
|  |  | (35) |
|  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  | (36) |
|  | in air |  |

Therefore, we can see from Fig. 3, (33) and (35) that,

Also from (32) and (34),

|  |  |  |
| --- | --- | --- |
|  |  | (37) |
|  | in air |  |

So we have the focal length being the reciprocal of the first Gaussian constant and also have,

|  |  |  |
| --- | --- | --- |
|  |  | (38) |

Furthermore, from (34) and (36), and (35) and (37) we have,

|  |  |  |
| --- | --- | --- |
|  |  | (39) |
|  |  | (40) |

Thus, if and are greater than unity, then it tells you the unit object/image planes lie outside the lense and provides and indication of the extent the unit planes lie beyond the lense. Usually, however and are actually less than unity and thus tell you that the unit planes are in the lens and indicate the fraction of the total focal lengths, and which lies outside the lens, and . Lens designer tend to use the term *back focus* for to indicate that it is different from the focal length .

Example: Asymmetric double convex lens (note: back surface curvature is negative as to left of lense)

=2

=0.5

Ray tracing: Refraction at , translation from to and then refraction at .

(focal length of lense)

(back focus)

, thus plane located 0.24 units right of

, thus plane located 0.12 units left of

The unit planes are located within the lens. This happens to be the case for all double convex lenses where thickness is not negligible.

Example: doublet – convex to plano-concave lens i.e. flipping previous example around and adding a plano-concave lense

=0.5

=1

=0.4

Convenient to format a ray tracing table (going down the table corresponding to tracing a ray from left to right in diagram above):

|  |  |  |
| --- | --- | --- |
|  |  |  |
| 1.0 |  |  |
|  | 1.5 | 0.5 |
| -2.0 |  |  |
|  | 1.632 | 0.4 |
|  |  |  |

(note: due to small difference in refractive indices not much refraction occurs at this interface. However ray bends slightly upwards due to + number in (1,3) element of R2)

Thus,

System matrix and Gaussian constants:

(i.e. H’ within the convex lens)

(i.e. H just outside surface 1 of convex lens)

This time drawing these points out:

=0.5

=0.4

Angular magnification

Assuming a ray is leaving the z-axis i.e. , at an inclination angle of , then from (30) we can obtain,

|  |  |  |
| --- | --- | --- |
|  |  | (41) |
|  |  | (42) |

That is ray at image point passes through z-axis at inclination angle . Thus we can define the angular magnification as,

|  |  |  |
| --- | --- | --- |
|  |  | (43) |

Therefore, if the media is the same on either side of the lens we have such that the angular magnification is always simply the reciprocal of the normal magnification of the lens,

|  |  |  |
| --- | --- | --- |
|  |  | (44) |

For the situation that, , additional nodal points and are defined on the z-axis and they correspond to the situation where or alternatively where we have unit angular magnification, . Thus, for rays leaving these nodal points the following must also be true from (43),

|  |  |  |
| --- | --- | --- |
|  |  | (45) |

From the definition of magnification in terms of the Gaussian constants (29), (43), and and on these nodal points, we can obtain,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (46) |

and,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (47) |

Seldom does one work with systems where the two media on either side are not the same. If equations (46) and (47) simply become equations (32) and (33), which tell you that the points of unit angular magnification, , coincide with for unit magnification.

The lensmaker’s equation and thin lens approximation

Recall from (22) that, . Using this and (36) and (37) we obtain,

|  |  |  |
| --- | --- | --- |
|  |  | (48) |

Therefore, in air we have,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (49) |

Which is referred to as the lensmaker’s equation as it relates the focal length of the lense with its geometry. For a thin lens we can say the second term on (49) is negligible (i.e. much smaller than or as typically ) giving the approximate formula,

|  |  |  |
| --- | --- | --- |
|  |  | (50) |

With the thin lens approximation we can also say that from (22), and using (32) and (33) that, . That is, the unit planes lie directly on the vertices of the lens with the z-axis. Thus, with the assumption of the thin lens approximation, the back focus is equivalent to the focal length of the lens:

If we use (since assuming air), the system matrix for a thin lens is,

|  |  |  |
| --- | --- | --- |
|  | (thin lenses) | (51) |

Example: compound magnifier – double convex lens followed by double concave lens (thin lens)

12 units

4

3

2

1

If object is located 60 units left () from convex lens find the image location and magnification?

(system matrix for convex)

(translation in air between lenses)

(system matrix for concave)

(system matrix for compound lens)

For system the Gaussian constants are,

Therefore, focal length for combination is and the back focus is

But,

Also,

.

Image is located 39.83 units to right of the second lense with magnification

Ray tracing from object of 10 units height above axis and ray leaving which is parallel to axis:

Thus at the image, ray passes 16.6 units below axis with angle of declination of 0.467 radians.

Determining location of cardinal planes and sketching in plus ray tracing from prev:

12 units

25.7 units

17.1 units

21.43 units

4.29 units

10 units

60 units

16.7 units

30 units

Note: generalisations to example

|  |  |  |
| --- | --- | --- |
|  | (for thin lenses) |  |

Element (1,2) to above system matrix for compound magnifier (see prev example).

|  |  |  |
| --- | --- | --- |
|  | In air,  (for thin lenses – this holds for compound magnifier) | (52) |

In general for thick lenses the following holds (derivation not shown here as it is obvious how to do):

|  |  |  |
| --- | --- | --- |
|  |  | (53) |

Image to object distance for thin lenses in air

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  |  |

Therefore object-to-image distance for thin lenses is (noting convention that left of lens is negative),

|  |  |  |
| --- | --- | --- |
|  |  | (54) |

Form of Gauss’s Law and magnification for thin lenses from object-to-image matrix

Using (29) and (37) we can find the object-to-image matrix for thin lenses as follows,

|  |  |  |
| --- | --- | --- |
|  |  |  |
|  |  | (55) |
|  | Can also show that,  (with convention that below axis is negative) | (56) |

Fun example to test understanding: Transparent sphere with one hemisphere with silvered reflecting surface.

Finding the system matrix ?

|  |  |  |
| --- | --- | --- |
| **r** | **n** | **t** |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |

, , (reflection at treated as a refraction into a material with refractive index , ,

Zoom lens example: varying magnification but maintaining object-to-image distance:

It is convenient to introduce the focal plane matrix for a single lens for this example,

|  |  |  |
| --- | --- | --- |
|  | Using, , , and we find,  Since, we find after multiplying out the above,  In air we have so, | (57) |
|  |  | (58) |

Thus for lay leaving and parallel at , ray has height at and as indicated by (2,2) element in . Likewise, if ray travelled the other direction and left parallel from then height is , giving as indicated by the (1,1) element in . Thus, the matrix we confirm displays the required behaviour.

We can determine the cardinal planes from the focal plane matrix for the single symmetrical double convex lens if we knew the thickness of the lens or the refractive power where the other can be determined knowing the focal length and using the Lensmaker’s equation (since where the refractive powers are equal due to symmetry of the lens), and computing the Gaussian constant c. Since, and because of the symmetry of the double convex lens . Thus, from the Gaussian constant the location of the focal planes can be determined and so can the unit planes through .

Now we include three lenses in air with separation between focal points , and respectively as shown in diagram below.

Series of focal plane and translation matrices for each element

Multiply matrices in order following arrow for ray tracing, for tracing a ray going from, left to right through series of lenses to find .

**Fig. 4**

For the first two lenses we can find the combined focal plane matrix,

|  |  |  |
| --- | --- | --- |
|  |  | (59) |

For the combined system consisting of only the first two lenses the cardinal planes can also be determined using the formulae (32)- (33), and (35)-(36), however again the Gaussian constants need to be known for each lens to find the Gaussian constants for the combined lens system. However, if we assume each lens behaves as a thin lens then we can find the system matrix for the two lenses by working backwards from the focal plane matrix by introducing minus translations by on right of two lens system and on the left of the system.

|  |  |  |
| --- | --- | --- |
|  |  |  |

Using (32) and (33), we can find the unit plane for the two lens system,

|  |  |  |
| --- | --- | --- |
|  |  | (60) |
|  |  | (61) |

Using (34) and (35) we can find the focal planes of the combined two lens system focal planes,

|  |  |  |
| --- | --- | --- |
|  |  | (62) |
|  |  | (63) |
|  |  | (64) |

For all three lenses using result (59) and symmetry of problem we find the complete focal plane matrix,

|  |  |  |
| --- | --- | --- |
|  |  | (65) |

Like for the doublet system if we assume individual matrices are thin we can determine system matrix for the triplet system using its focal plane matrix and subtracting out rays through use of a translation matrices up to the lens vertices,

|  |  |  |
| --- | --- | --- |
|  |  | (66) |

Thus we can find the location of the cardinal planes in the same manner for the three lens system,

|  |  |  |
| --- | --- | --- |
|  |  | (67) |
|  |  | (68) |
|  |  | (69) |
|  |  | (70) |

The sum is

|  |  |  |
| --- | --- | --- |
|  |  | (71) |

From Fig. 4 we want to maintain the distance between the focal planes of the combined three lens system while holding the first and third lens stationary as the second lens is moved. This can only happen if the sum is constant. The simplest way of doing this is by making them numerically equal such that . From (71) this can be done by making, and (i.e. making the first and last lens identical), which make the numerator zero, as shown in figure below.

Maintaining distance between focal planes is only the first step however. The focal length for the triplet system where distance between focal planes are maintained is,

The focal length therefore varies with separation , and since the focal planes remain fixed (F and F’) the unit planes (H and H’) must also vary with , and therefore the object-to-image distance will also vary slightly and the image will be out of focus. To study this in greater detail will shall consider the following example:

= 20

100

60

For system of thin lenses the system matrix is,

From Eq. (31) we have,

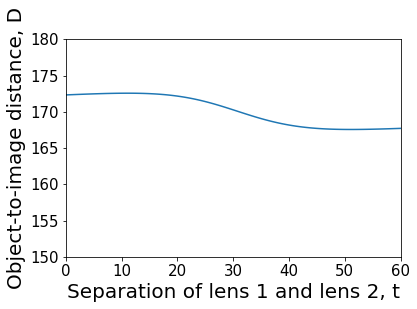
(in air)

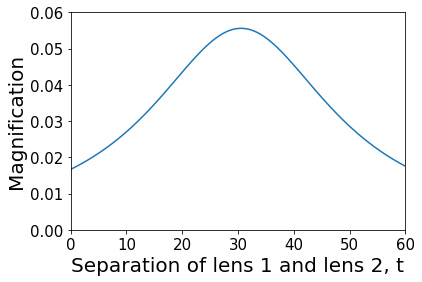
Where, the object to distance image can be computed by the elements of the system matrix () through,

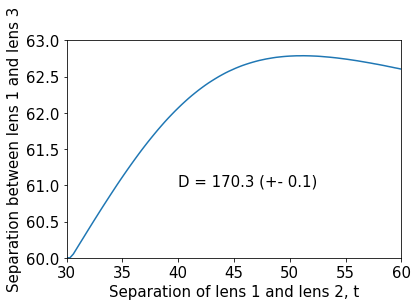
Similarly the magnification can be computed by,

On the next page the change in object-to-image distance and magnification with the lens separation, is shown.

In order, to maintain the object-to-image distance and thus achieve focus at a fixed image plane, as lens 2 shifted, lens 3 can also be shifted slightly simultaneously to correct for the defocusing. For instance if initially lens 2 is at 30 units to the right of lens 1 and we adjust magnification by shifting lens 2 towards lens 3. The amount lens 3 needs to shift right to correct for the defocusing, i.e. to maintain object-to-image distance by + or - 0.1 units, is shown in the figure in the next page. This was computed numerically with Python code.







To complete chapter 1 on introducing ray tracing and the paraxial approximation, we shall look at a slightly more complicated optical system shown below. Such a lens system is known as a *Tessar* and is used in moderately priced cameras.

|  |  |  |
| --- | --- | --- |
|  |  |  |
| 1.628 |  |  |
|  | 1.6116 | 0.357 |
| -27.57 |  |  |
|  | 1 | 0.189 |
| -3.457 |  |  |
|  | 1.6053 | 0.081 |
| 1.582 |  |  |
|  | 1 | 0.325 |
|  |  |  |
|  | 1.5123 | 0.217 |
| 1.920 |  |  |
|  | 1.6116 | 0.396 |
| -2.4 |  |  |

This example, is computed in Python. A library called paraxialRayApproximation.py with functions to compute the refraction, translation, and subsequently the system matrices were generated.

**S21:**

Example procedure to compute system matrix for first lens (leftmost):

**Complete system matrix is then by determining system,translation and refractive matrices for entire system can then be computed by,**

**And the cardinal planes and focal length can then also be determined by a function created in this python using the above, a,b,c and d Gaussian constants and the following formulae:**

(Shown in above figure)

**What if we place an object 100 units in front of lens (), then what would be the position of the image plane, magnification and object-to-image distance?**

Programming the formula,

we can determine the magnification from the object position.

Then, using,

And then,

We can determine the position of the image plane and the object-to-image distance.

Giving, , , .